On contact structures and open books

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Novosibirsk, August 24th, 2015

Contact structures

Three-manifolds have trivial tangent bundles. Let ξ be a plane field defined by a Pfaffian equation $\{\alpha = 0\}$.

Theorem (Frobenius theorem)

A plane field is integrable if and only if $\alpha \wedge d\alpha \equiv 0$.

The integration is usually called a foliation. For example, the three-sphere has the Reeb foliation. In fact every codimension-one foliation of S^3 has a Reeb component.

On the other extreme, a plane field can be nowhere integrable.

Definition

If $\alpha \wedge d\alpha$ does not vanish, that is, $\alpha \wedge d\alpha$ is nowhere zero, then ξ is nowhere integrable and is called a *contact structure*.

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Overtwisted contact structures

Definition

A contact structure ξ is called *overtwisted* if there exists an embedded disk $D \subset M$ such that the characteristic foliation D_ξ contains one closed leaf C and exactly one singular point $p \in D$ inside C. Otherwise it is called tight.

The contact structure $\xi_4 = \ker(\cos(r)dz + r\sin(r)d\theta)$ is horizontal when $r = k\pi$. A neighborhood of $z = 0, r \leq \pi$ looks like

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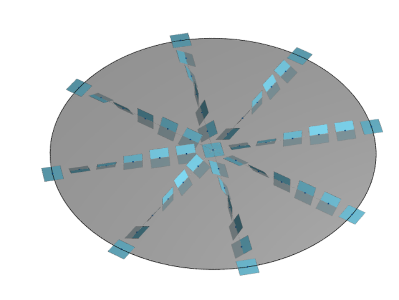

The dichotomy: Tightness vs overtwistedness

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A contact structure is *tight* if for any embedded disc $D \subset M$, the characteristic foliation D_ξ contains no limit cycles.

Theorem (Bennequin, 1983)

S^3 with the standard contact structure is tight.

Theorem (Eliashberg, 1992)

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Homotopy obstructions

Overtwisted contact structures can be obtained via Lutz twists.

Theorem (Eliashberg, 1989)

The isotopy classification of overtwisted contact structures on a closed three-manifold coincide with their homotopy classification as tangent plane fields.

So overtwisted contact structures have extreme flexibility.

However, for tight contact structures, we have the Bennequin-Eliashberg inequality

$$\langle e(\xi), [F] \rangle \leq -\chi(F), \quad (F \text{ not spherical})$$

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Open book decomposition

Definition

An *open book decomposition* of a three-manifold Y is a pair (B, π) where

- B is an oriented link in Y, called the binding of the open book, and
- $\pi : Y \setminus B \to S^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset M$ and $\partial\Sigma_\theta = B$ for all $\theta \in S^1$. The surface $\Sigma = \Sigma_\theta$, for any θ, is called the page of the open book.

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Theorem (Thurston-Winkelnkemper, 1975)

For a three-manifold Y with an open book decomposition, one can construct a contact structure on Y.

Corollary

Every oriented three-manifold has a contact structure.

Definition

A contact structure ξ is *compatible with* or *supported by* an open book (B, π) of Y if ξ can be isotoped through contact structures so that there is a contact one-form α for ξ such that

- $d\alpha$ is a positive area form on each page Σ_θ of the open book,
- $\alpha > 0$ on B.

The contact structure constructed by Thurston-Winkelnkemper is supported by the open book.

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Intuitively, an open book (B, π) supports a contact structure ξ, if the binding B is positively transverse to ξ and on the complement of B the contact planes ξ can be isotoped to be arbitrarily close to the pages of the open book while keeping B transverse.

Theorem (Giroux, 2002)

Let Y be a compact oriented 3-manifold. Then there is a bijection between the set of oriented contact structures on Y up to isotopy and the set of open book decompositions of M up to positive stabilization.

Positive stabilization consists of modifying the page by adding a 2-dimensional 1-handle and modifying the monodromy by adding a positive Dehn twist along a curve that runs over that handle exactly once.

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Hence there is no homotopy-theoretic obstruction to a contact structure admitting a planar open book.

However, not every contact three-manifold is planar:

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Let X be a symplectic filling of a planar contact 3-manifold (Y, ξ). Then $b_2^+(X) = b_2^0(X) = 0$ and X has connected boundary. Furthermore, if Y is an integral homology sphere, then the intersection form of X is diagonalizable.

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This implies the following for a contact three-manifold (Y, ξ):

- If $c^+(\xi) \neq 0$ and, for the associated Spin^c structure $\mathfrak{s}(\xi)$, $c_1(\mathfrak{s}(\xi))$ is not a torsion class, then (Y, ξ) cannot be planar.
- If $c_1(\mathfrak{s}(\xi)) = 0$ and (Y, ξ) admits a Stein filling (X, J) such that $c_1(X, J) \neq 0$, then (Y, ξ) cannot be planar.
- For a Legendrian knot L in S^3 with standard contact structure with vanishing Thurston-Bennequin invariant, the contact manifold (Y, ξ) obtained from the Legendrian surgery on L is not planar.

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Using holomorphic curves and techniques from symplectic field theory, Niederkrüger and Wendl showed that a weakly fillable but not Stein fillable contact structure is nonplanar.

Theorem (Niederkrüger-Wendl, 2010)

Every weak filling of a planar contact three-manifold is symplectically deformation equivalent to a blow-up of a Stein filling.

This implies, for instance, a weakly symplectic fillable contact structure with 2π Giroux torsion is nonplanar.

Question

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Giroux introduced the notion of torsions.

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A contact manifold (Y, ξ) has *positive $n\pi$ torsion* ($n \in \mathbb{N}$) if it admits a contact embedding $(T^2 \times [0, n], \xi_*) \hookrightarrow (Y, \xi)$, where $\xi_* = \ker(\sin(\pi z)dx + \cos(\pi z)dy)$ and (x, y, z) are the coordinates on $T^2 \times [0, n] \cong \mathbb{R}^2/\mathbb{Z}^2 \times [0, n]$.

Goal: investigate Giroux torsion in planar contact three-manifold.

Theorem (Li-W)

There exist planar tight contact three-manifolds with nontrivial 2π torsion.

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A planar tight contact manifold with positive torsion

Let $Y = M\left(S^2; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ be the Seifert fibred space with base orbifold S^2 and 4 singular fibers.

$$Y \cong M\left(D^2; \frac{1}{2}, -\frac{1}{2}\right) \cup_{T^2} M\left(D^2; \frac{1}{2}, -\frac{1}{2}\right)$$

We will contruct a planar tight contact structrue on Y with positive 2π torsion along T.

On $M(D^2, \frac{1}{2}, -\frac{1}{2})$, we construct the open book (P, φ) which supports a tight contact structure

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Let $Y = M\left(S^2; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ be the Seifert fibred space with base orbifold S^2 and 4 singular fibers.

$$Y \cong M\left(D^2; \frac{1}{2}, -\frac{1}{2}\right) \cup_{T^2} M\left(D^2; \frac{1}{2}, -\frac{1}{2}\right)$$

We will contruct a planar tight contact structrue on Y with positive 2π torsion along T.
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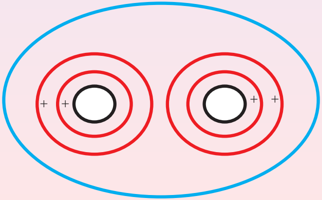

Gluing torsion

The following is the relative open book decomposition of

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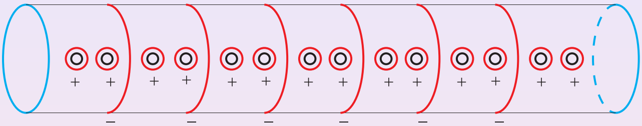

Gluing these relative open books, we obtain a planar open book supporting (Y, ξ):

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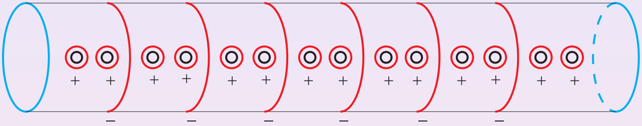

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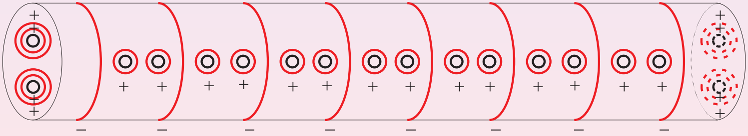

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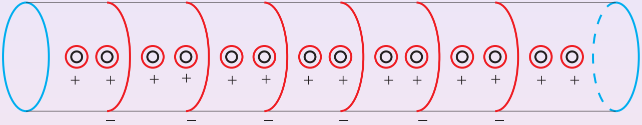

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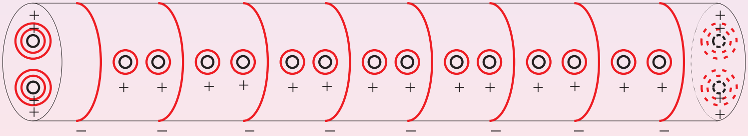

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Remark

We remark that the Giroux torsion is separating in our example. It will be interesting to ask whether a planar tight contact manifold can have a non-separating torsion.

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Can planar tight contact three-manifold have Giroux torsion of arbitrarily large size?

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Support genus of Legendrian knots

Definition (Etnyre)

Let L be a Legendrian knot in a contact three-manifold (M, ξ), the *support genus* of L, denoted by $sg(L)$, is the minimal genus of a page among all open book decompositions of M supporting ξ such that L sits on a page of the open book and the framings given by ξ and given by the page coincide.

Question (Onaran)

Does every Legendrian knot in (S^3, ξ_{std}) with negative Thurstion-Bennequin invariant have support genus zero?

Theorem (Li-W)

Suppose $k \geq 1$. Let L be a Legendrian $(2, 2k+1)$ torus knot in (S^3, ξ_{std}) with nonnegative Thurston-Bennequin invariant, then $sg(L) = 1$.

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Idea of proof

- Construct a fibered link with a Thurston norm minimizing Seifert surface which contains the interested knot.
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Thank you!